

SEMICANONICAL BASIS GENERATORS OF THE CLUSTER ALGEBRA OF TYPE $A_1^{(1)}$

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1. INTRODUCTION

The (coefficient-free) cluster algebra \mathcal{A} of type $A_1^{(1)}$ is a subring of the field $\mathbb{Q}(x_1, x_2)$ generated by the elements x_m for $m \in \mathbb{Z}$ satisfying the recurrence relations

$$(1) \quad x_{m-1}x_{m+1} = x_m^2 + 1 \quad (m \in \mathbb{Z}).$$

This is the simplest cluster algebra of infinite type; it was studied in detail in [2, 6]. Besides the generators x_m (called *cluster variables*), \mathcal{A} contains another important family of elements s_0, s_1, \dots defined recursively by

$$(2) \quad s_0 = 1, \quad s_1 = x_0x_3 - x_1x_2, \quad s_n = s_1s_{n-1} - s_{n-2} \quad (n \geq 2).$$

As shown in [2, 6], the elements s_1, s_2, \dots together with the *cluster monomials* $x_m^p x_{m+1}^q$ for all $m \in \mathbb{Z}$ and $p, q \geq 0$, form a \mathbb{Z} -basis of \mathcal{A} referred to as the *semicanonical basis*.

As a special case of the *Laurent phenomenon* established in [3], \mathcal{A} is contained in the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$. In particular, all x_m and s_n can be expressed as integer Laurent polynomials in x_1 and x_2 . These Laurent polynomials were explicitly computed in [2] using their geometric interpretation due to P. Caldero and F. Chapoton [1]. As a by-product, there was given a combinatorial interpretation of these Laurent polynomials, which can be easily seen to be equivalent to the one previously obtained by G. Musiker and J. Propp [5].

The purpose of this note is to give short, self-contained and completely elementary proofs of the combinatorial interpretation and closed formulas for the Laurent polynomial expressions of the elements x_m and s_n .

2. RESULTS

We start by giving an explicit combinatorial expression for each x_m and s_n , in particular proving that they are Laurent polynomials in x_1 and x_2 with positive integer coefficients. By an obvious symmetry of relations (1), each element x_m is obtained from x_{3-m} by the automorphism of the ambient field $\mathbb{Q}(x_1, x_2)$ interchanging x_1 and x_2 . Thus, we restrict our attention to the elements x_{n+3} for $n \geq 0$.

Date: June 29, 2006.

2000 *Mathematics Subject Classification.* Primary 16S99.

Research supported by NSF (DMS) grant # 0500534 and by a Humboldt Research Award.

Following [2, Remark 5.7] and [4, Example 2.15], we introduce a family of *Fibonacci polynomials* $F(w_1, \dots, w_N)$ given by

$$(3) \quad F(w_1, \dots, w_N) = \sum_D \prod_{k \in D} w_k,$$

where D runs over all *totally disconnected* subsets of $\{1, \dots, N\}$, i.e., those containing no two consecutive integers. In particular, we have

$$F(\emptyset) = 1, \quad F(w_1) = w_1 + 1, \quad F(w_1, w_2) = w_1 + w_2 + 1.$$

We also set

$$(4) \quad f_N = x_1^{-\lfloor \frac{N+1}{2} \rfloor} x_2^{-\lfloor \frac{N}{2} \rfloor} F(w_1, \dots, w_N)|_{w_k = x_{\langle k+1 \rangle}^2},$$

where $\langle k \rangle$ stands for the element of $\{1, 2\}$ congruent to k modulo 2. In view of (3), each f_N is a Laurent polynomial in x_1 and x_2 with positive integer coefficients. In particular, an easy check shows that

$$(5) \quad f_0 = 1, \quad f_1 = \frac{x_2^2 + 1}{x_1} = x_3, \quad f_2 = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2} = s_1.$$

Theorem 2.1. [2, Formula (5.16)] *For every $n \geq 0$, we have*

$$(6) \quad s_n = f_{2n}, \quad x_{n+3} = f_{2n+1}.$$

In particular, all x_m and s_n are Laurent polynomials in x_1 and x_2 with positive integer coefficients.

Using the proof of Theorem 2.1, we derive the explicit formulas for the elements x_m and s_n .

Theorem 2.2. [2, Theorems 4.1, 5.2] *For every $n \geq 0$, we have*

$$(7) \quad x_{n+3} = x_1^{-n-1} x_2^{-n} (x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q} x_2^{2r});$$

$$(8) \quad s_n = x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r}.$$

3. PROOF OF THEOREM 2.1

In view of (3), the Fibonacci polynomials satisfy the recursion

$$(9) \quad F(w_1, \dots, w_N) = F(w_1, \dots, w_{N-1}) + w_N F(w_1, \dots, w_{N-2}) \quad (N \geq 2).$$

Substituting this into (4) and clearing the denominators, we obtain

$$(10) \quad x_{\langle N \rangle} f_N = f_{N-1} + x_{\langle N-1 \rangle} f_{N-2} \quad (N \geq 2).$$

Thus, to prove (6) by induction on n , it suffices to prove the following identities for all $n \geq 0$ (with the convention $s_{-1} = 0$):

$$(11) \quad x_1 x_{n+3} = s_n + x_2 x_{n+2};$$

$$(12) \quad x_2 s_n = x_{n+2} + x_1 s_{n-1}.$$

We deduce (11) and (12) from (2) and its analogue established in [6, formula (5.13)]:

$$(13) \quad x_{m+1} = s_1 x_m - x_{m-1} \quad (m \in \mathbb{Z}).$$

(For the convenience of the reader, here is the proof of (13). In view of (2) and (1), we have

$$s_1 = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2} = \frac{x_1 + x_3}{x_2} = \frac{x_0 + x_2}{x_1}.$$

By the symmetry of the relations (1), this implies that $s_1 = (x_{m-1} + x_{m+1})/x_m$ for all $m \in \mathbb{Z}$, proving (13).)

We prove (11) and (12) by induction on n . Since both equalities hold for $n = 0$ and $n = 1$, we can assume that they hold for all $n < p$ for some $p \geq 2$, and it suffices to prove them for $n = p$. Combining the inductive assumption with (2) and (13), we obtain

$$\begin{aligned} x_1 x_{p+3} &= x_1 (s_1 x_{p+2} - x_{p+1}) \\ &= s_1 (s_{p-1} + x_2 x_{p+1}) - (s_{p-2} + x_2 x_p) \\ &= (s_1 s_{p-1} - s_{p-2}) + x_2 (s_1 x_{p+1} - x_p) \\ &= s_p + x_2 x_{p+2}, \end{aligned}$$

and

$$\begin{aligned} x_2 s_p &= x_2 (s_1 s_{p-1} - s_{p-2}) \\ &= s_1 (x_{p+1} + x_1 s_{p-2}) - (x_p + x_1 s_{p-3}) \\ &= (s_1 x_{p+1} - x_p) + x_1 (s_1 s_{p-2} - s_{p-3}) \\ &= x_{p+2} + x_1 s_{p-1}, \end{aligned}$$

finishing the proof of Theorem 2.1.

4. PROOF OF THEOREM 2.2

Formulas (7) and (8) follow from (11) and (12) by induction on n . Indeed, assuming that, for some $n \geq 1$, formulas (7) and (8) hold for all the terms on the right hand side of (11) and (12), we obtain

$$\begin{aligned} x_{n+3} &= x_1^{-1} (s_n + x_2 x_{n+2}) \\ &= x_1^{-n-1} x_2^{-n} \left(\sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r} \right. \\ &\quad \left. + (x_2^{2(n+1)} + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2(r+1)}) \right) \\ &= x_1^{-n-1} x_2^{-n} (x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} (\binom{n-q}{r} + \binom{n-q}{r-1}) x_1^{2q} x_2^{2r}) \\ &= x_1^{-n-1} x_2^{-n} (x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q} x_2^{2r}), \end{aligned}$$

and

$$\begin{aligned}
s_n &= x_2^{-1}(x_{n+2} + x_1 s_{n-1}) \\
&= x_1^{-n} x_2^{-n} (x_2^{2n} + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r} \\
&\quad + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-1-q}{r} x_1^{2(q+1)} x_2^{2r}) \\
&= x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \left(\binom{n-1-r}{q} + \binom{n-1-r}{q-1} \right) \binom{n-q}{r} x_1^{2q} x_2^{2r} \\
&= x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r},
\end{aligned}$$

as desired.

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